GAUSS-JORDAN STUFF

Gaussian elimination.

Here is the basic outline of the Gaussian elimination process for an augmented matrix. Let A be the augmented matrix of a linear system of variables.

(1) Interchange rows in A so that the leftmost nonzero entry of A is in the first row.
(2) Add multiples of the first row to lower rows so that there are only zeros below the pivot of the first row.
(3) Now... *forget about the first row and first column of* A! Cover them up or something. Now you have a smaller matrix. Call it B. Repeat steps 1,2, and 3 with B until you run out of matrix!

We'll see an example later.

Echelon form.

The idea with Gaussian elimination is that it gets the matrix into echelon form. Here's a reminder of the definition. A matrix is in echelon form if the following are true:

(1) Every leading term is to the left of the leading term in the lower rows.
(2) All zero rows are at the bottom.

But wait—there's more!

Gauss-Jordan elimination.

Here is the basic outline of the Gauss-Jordan elimination process for an augmented matrix. Again, let A be the augmented matrix of a linear system of variables.

(1) Do Guassian elimination first—now the matrix is in echelon form.
(2) Multiply each row with a pivot entry by the reciprocal of the pivot so that the pivot now equals 1.
(3) Find the row with the rightmost pivot. Add multiples of that row to the above rows so that only zeros remain above the pivot.
(4) Move to the *next* rightmost pivot, and repeat steps 3 and 4 until you run out of pivots!

We'll see an example later.

Reduced row echelon form.

The idea with Guass-Jordan elimination is that it gets the matrix into its *unique* reduced (row) echelon form. Here's a reminder of the definition. A matrix is in reduced (row) echelon form if the following are true:

(1) The matrix is in echelon form.
(2) Every pivot equals 1.
(3) The only nonzero entry in each pivot column is the pivot itself.

An example.

Let's consider the augmented matrix

$$\begin{bmatrix} 0 & 1 & 3 & 6 & 11 \\ 2 & -1 & 2 & 0 & 7 \\ 2 & -2 & 0 & -5 & 0 \\ -6 & 5 & -1 & 11 & 0 \end{bmatrix}$$

and perform Guassian elimination on it. Step 1:

$$\begin{bmatrix} 0 & 1 & 3 & 6 & 11 \\ 2 & -1 & 2 & 0 & 7 \\ 2 & -2 & 0 & -5 & 0 \\ -6 & 5 & -1 & 11 & 0 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 2 & -2 & 0 & -5 & 0 \\ -6 & 5 & -1 & 11 & 0 \end{bmatrix}.$$

Step 2:

$$\begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 2 & -2 & 0 & -5 & 0 \\ -6 & 5 & -1 & 11 & 0 \end{bmatrix} \underset{\sim}{R_3 - R_1 \to R_3} \begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 0 & -1 & -2 & -5 & -7 \\ -6 & 5 & -1 & 11 & 0 \end{bmatrix} \underset{\sim}{R_4 + 3R_1 \to R_4} \begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 0 & -1 & -2 & -5 & -7 \\ 0 & 2 & 5 & 11 & 21 \end{bmatrix}.$$

Step 3: forget about the first row and column and keep going! Another round of step 1:

$$\begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 0 & -1 & -2 & -5 & -7 \\ 0 & 2 & 5 & 11 & 21 \end{bmatrix}$$

already has the next pivot in the correct place. Step 2:

$$\begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 0 & -1 & -2 & -5 & -7 \\ 0 & 2 & 5 & 11 & 21 \end{bmatrix} \underset{\sim}{R_3 + R_2 \to R_3} \begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 2 & 5 & 11 & 21 \end{bmatrix} \underset{\sim}{R_4 - 2R_2 \to R_4} \begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix}.$$

Step 3: now, forget about the *second* row and *second* column and keep going! Another round of step 1:

$$\begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix}$$

already has the pivot in the correct place. Step 2:

$$\begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix} \underset{\sim}{R_4 + R_3 \to R_4} \begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Step 3: forget about the *third* row and *third* column, and keep going! Only $\begin{pmatrix} 0 & 3 \end{pmatrix}$ remains in the bottom right corner, and there is nothing left to do. We have just completed Gaussian elimination, which is also step 1 of Gauss-Jordan elimination!

The matrix is therefore in echelon form Let's finish the Gauss-Jordan elimination process. Step 2:

$$\begin{bmatrix} 2 & -1 & 2 & 0 & 7 \\ 0 & 1 & 3 & 6 & 11 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \overset{\frac{1}{2}R_1 \to R_1}{\sim} \begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 & \frac{7}{2} \\ 0 & 1 & 3 & 6 & 11 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \overset{\frac{1}{3}R_4 \to R_4}{\sim} \begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 & \frac{7}{2} \\ 0 & 1 & 3 & 6 & 11 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 3: the first pivot is the 1 in row 4. Observe

$$\begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 & \frac{7}{2} \\ 0 & 1 & 3 & 6 & 11 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 1 & 3 & 6 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

by subtracting multiples of row 4 from the rows above. Step 4: the next pivot is the 1 in row 3, column 3, so we repeat step 3 as follows:

$$\begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 1 & 3 & 6 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{R_2 - 3R_3 \to R_2}{\sim} \begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{R_1 - R_3 \to R_1}{\sim} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 4: the next pivot is the 1 in the second column—time for another round of step 3!

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{R_1 + \frac{1}{2}R_2 \to R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 4: the *only* remaining pivot is the top-left 1 in the matrix, and there is no need to do step 3 again. This completes Gauss-Jordan elimination, and the matrix above is in reduced (row) echelon form!

(Note: we encountered no zero rows. Fortunately, the procedure I wrote down above ensures that all the zero rows will show up at the bottom of the matrix in the end.)

Notice that this procedure has reduced the linear system

$$\begin{aligned} x_2 + 3x_3 + 6x_4 &= 11 \\ 2x_1 - x_2 + 2x_3 &= 7 \\ 2x_1 - 2x_2 - 5x_4 &= 0 \\ -6x_1 + 5x_2 - x_3 + 11x_4 &= 0 \end{aligned}$$

to the (inconsistent) linear system

$$\begin{aligned} x_1 + \tfrac{1}{2}x_4 &= 0 \\ x_2 + 3x_4 &= 0 \\ x_3 + x_4 &= 0 \\ 0 &= 1. \end{aligned}$$