Cycle type factorizations in $\mathrm{GL}_n\mathbb{F}_q$

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"You don't start out writing good stuff. You start out writing crap and thinking it's good stuff, and then gradually you get better at it."

- Octavia Butler

"I've proven something original, but I would still call it pretty trivial."

- Dan Snaith

"If I had 53 minutes to spend as I liked, I should walk at my leisure toward a spring of fresh water."

- *The Little Prince*, Antoine de Saint-Exupéry

Introduction

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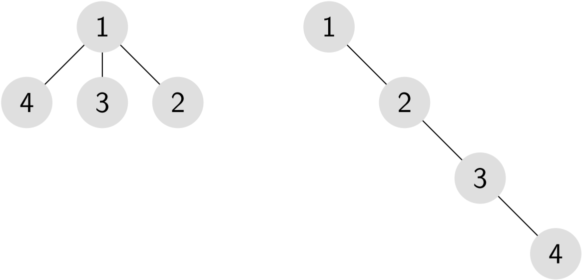

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Example

The transpositions are $\mathcal{C}_{(2,1^{n-2})}$, and $\deg \chi^\lambda = \chi^\lambda_{(1^n)}$.

Main inspiration

Definition

For all $n, k \in \mathbb{N}$ and $\mu \vdash n$, define

$$g_{k,\mu} = \#\{(t_1, \ldots, t_k) \in \mathcal{C}_{(n)}^k : t_1 \cdots t_k \in \mathcal{C}_\mu\}.$$

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Theorem (Stanley)

For all $n, k \in \mathbb{N}$ and $\mu \vdash n$, we have

$$\frac{g_{k,\mu}}{\#\mathcal{C}_\mu} = \frac{(n-1)!^{k-1}}{n} \sum_{r=0}^{n-1} \frac{(-1)^{rk} \chi_\mu^{(n-r,1^r)}}{\binom{n-1}{r}^{k-1}}.$$

Application to Hurwitz theory:

$$H_{3(n-1)\xrightarrow{n}0}\big((n),(n),(n)\big) = \frac{g_{3,(1^n)}}{n!} = \begin{cases} 0 & n \text{ even,} \\ \frac{2(n-1)!}{n(n+1)} & n \text{ odd.} \end{cases}$$

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- *Riemann Surfaces and Algebraic Curves: A First Course in Hurwitz Theory* -Cavalieri and Miles

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Theorem (Lewis-Reiner-Stanton)

For all $n \geq 2$ and prime powers q, the number of ordered n-tuples of reflections in $\mathrm{GL}_n\mathbb{F}_q$ whose product is a fixed, arbitrary Singer cycle equals $(q^n - 1)^{n-1}$.

Theorem (Lewis-Morales)

Fix a Singer cycle $c \in \mathrm{GL}_n\mathbb{F}_q$. Let $a_{r,s}(q)$ be the number of pairs (u, v) of elements of $\mathrm{GL}_n\mathbb{F}_q$ such that u has fixed-space dimension r, v has fixed-space dimension s, and $c = u \cdot v$. Then

$$\frac{1}{\#\mathrm{GL}_n\mathbb{F}_q} \sum_{r,s\geq 0} a_{r,s}(q) \cdot x^r y^s = \frac{(x;q^{-1})_n}{(q;q)_n} + \frac{(y;q^{-1})_n}{(q;q)_n} +$$

$$\sum_{\substack{0\leq t,u\leq n-1 \\ t+u\leq n}} q^{tu-t-u} \frac{[n-t-1]!_q \cdot [n-u-1]!_q}{[n-1]!_q \cdot [n-t-u]!_q} \frac{(q^n - q^t - q^u + 1)}{(q-1)}$$

$$\times \frac{(x;q^{-1})_t}{(q;q)_t} \frac{(y;q^{-1})_u}{(q;q)_u}.$$

Factorization results

Analogous structures

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cycle type	???

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A quote from Stong:

The analog of a cycle in $\pi \in S_d$ of length m is seen to be a polynomial $p(Z)$ of degree m that divides $\mathrm{char}(\alpha; Z)$

Definition (cycle type)

Suppose $g \in \mathrm{GL}_n\mathbb{F}_q$ has characteristic polynomial f, which factors into irreducibles as $f = f_1 \cdots f_\ell$ with weakly decreasing degrees. Define its *cycle type* to be

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Philosophy: $\mathcal{T}_\mu(q)$ is a q-analogue of $\mathcal{C}_\mu$.

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Corollary (Stong, see also Kung, Lehrer, Fulman)

As $q \to \infty$, an arbitrarily large proportion of $\mathrm{GL}_n\mathbb{F}_q$ elements have no repeated factors in their characteristic polynomial.

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The lattice of stable subspaces of $g \in \mathrm{GL}_n\mathbb{F}_q$ is Boolean if and only if g has no repeated factors in its characteristic polynomial.

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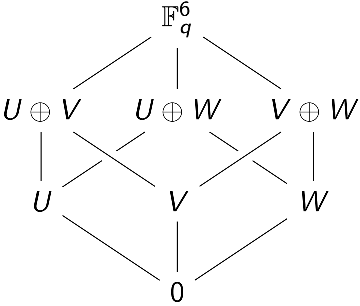

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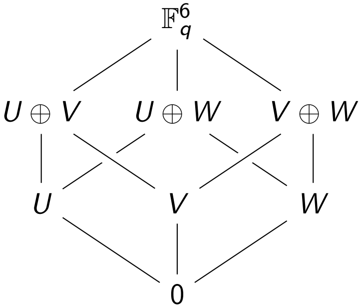

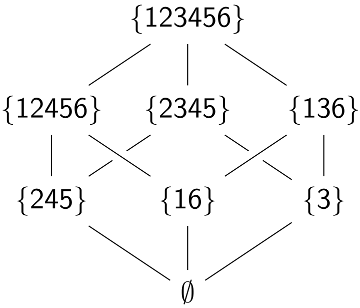

Definition

An element $g \in \mathrm{GL}_n\mathbb{F}_q$ is called *regular semisimple* if the irreducible factors of its characteristic polynomial are distinct.

Notation

For all $n \in \mathbb{N}$, $\mu \vdash n$, and prime powers q, define

$$\mathcal{T}_\mu^\square(q) = \{g \in \mathcal{T}_\mu(q) : g \text{ is regular semisimple}\}.$$

Philosophy: $\mathcal{T}_\mu^\square(q)$ is also a q-analogue of $\mathcal{C}_\mu$.

Note: $\mathcal{T}^{\square}_{(n)}(q) = \mathcal{T}_{(n)}(q) =$ the *regular elliptic* elements.

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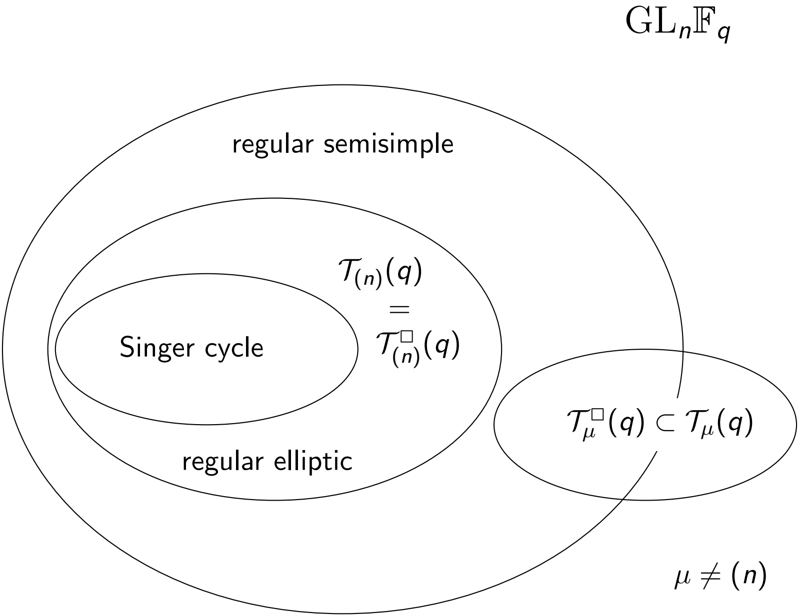
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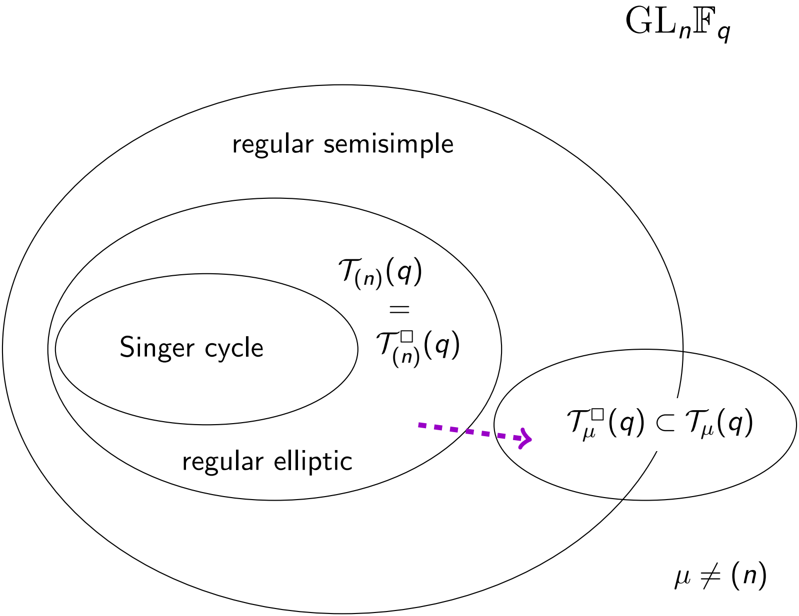
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Main result

Theorem

For all $n, k \in \mathbb{N}$ *with* $n > 2$*, all prime powers* q*, and all* $\mu \vdash n$ *with* $m_1(\mu) = 1$*, we have*

$$g^{\square}_{k,\mu}(q) = \frac{\#\mathcal{T}_{(n)}(q)^k \cdot \#\mathcal{T}^{\square}_{\mu}(q)}{\#\mathrm{GL}_n\mathbb{F}_q} \cdot \sum_{r=0}^{n-1} \frac{(-1)^{rk}\chi_{\mu}^{(n-r,1^r)}}{\left(q^{\binom{r+1}{2}} \cdot \begin{bmatrix} n-1 \\ r \end{bmatrix}_q\right)^{k-1}}.$$

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Compare to a rephrasing of Stanley's result:

$$g_{k,\mu} = \frac{\#\mathcal{C}^k_{(n)} \cdot \#\mathcal{C}_{\mu}}{\#\mathfrak{S}_n} \cdot \sum_{r=0}^{n-1} \frac{(-1)^{rk}\chi_{\mu}^{(n-r,1^r)}}{\binom{n-1}{r}^{k-1}}.$$

Corollary

Under the previous hypotheses ($m_1(\mu) = 1$), we have

$$\lim_{q \to 1} \frac{g^{\square}_{k,\mu}(q)/\#\mathcal{T}_{(n)}(q)^k}{\#\mathcal{T}^{\square}_{\mu}(q)/\#\mathrm{GL}_n\mathbb{F}_q} = \frac{g_{k,\mu}/\#\mathcal{C}^k_{(n)}}{\#\mathcal{C}_\mu/\#\mathfrak{S}_n}.$$

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Theorem

For all $n, k \in \mathbb{N}$ with $k \geq 2$ and all $\mu \vdash n$, we have

$$\lim_{q \to \infty} \frac{g^{\square}_{k,\mu}(q)/\#\mathcal{T}_{(n)}(q)^k}{\#\mathcal{T}^{\square}_{\mu}(q)/\#\mathrm{GL}_n\mathbb{F}_q} = 1.$$

The same holds without the $\square$.

Corollary (to main result)

For all $n, k \in \mathbb{N}$ with $n > 2$ and all prime powers q, we have

$$g_{k,(n-1,1)}(q) = \frac{\#\mathcal{T}_{(n)}(q)^k \cdot \#\mathcal{T}_{(n-1,1)}(q)}{\#\mathrm{GL}_n\mathbb{F}_q} \cdot \left(1 + \frac{(-1)^{nk-n-k}}{q^{\binom{n}{2}(k-1)}}\right).$$

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For all $n, k \in \mathbb{N}$ and prime powers q, we have a closed formula for

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Unsure how to compare to

$$g_{k,(n)} = \frac{\#\mathcal{C}_{(n)}^{k+1}}{\#\mathfrak{S}_n} \sum_{r=0}^{n-1} \left(\frac{(-1)^r}{\binom{n-1}{r}} \right)^{k-1}.$$

$$P_{n,k}(q) = \frac{1}{\#\mathrm{GL}_n\mathbb{F}_q}\left(\frac{(-1)^n\#\mathrm{GL}_n\mathbb{F}_q}{n(q^n-1)}\right)^k$$

$$\deg_{n,d,r}(q) = q^{d\binom{r+1}{2}} \cdot \frac{\prod_{i=1}^{n}(q^i-1)}{\prod_{j=1}^{n/d}(q^{jd}-1)} \cdot \begin{bmatrix} n/d-1 \\ r \end{bmatrix}_{q^d}$$

$$D_{n,k,d}(q) = \sum_{r=0}^{\frac{n}{d}-1}(-1)^{rk}\deg_{n,d,r}(q)^{2-k}$$

$$C_{n,k,c}(q) = \sum_{s_1,\ldots,s_k|n}\frac{(q^n-1)\prod_{i=1}^{k}\left[(q^{s_i}-1)\mu(n/s_i)\right]}{\mathrm{lcm}_{\mathbb{Z}}\left(\frac{q^n-1}{q^c-1}, q^{s_1}-1, \ldots, q^{s_k}-1\right)}$$

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$$\boxed{\frac{g_{k,(n)}(q)}{P_{n,k+1}(q)} = \sum_{d|n}\frac{d^k}{(-1)^{n(k+1)/d}}D_{n,k+1,d}(q)\sum_{c|d}\boldsymbol{\mu}\left(\frac{d}{c}\right)C_{n,k+1,c}(q)}$$

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Some progress:

Theorem

For all even prime powers q and $n, k \in \mathbb{N}$ with n odd, we have

$$g^{\square}_{k,\mu}(q) = \frac{\#\mathcal{T}_{(n)}(q)^k \cdot \#\mathcal{T}^{\square}_{\mu}(q)}{\#\mathrm{GL}_n\mathbb{F}_q} \cdot \sum_{r=0}^{n-1} \frac{(-1)^{rk}\chi_{\mu}^{(n-r,1^r)}}{\left(q^{\binom{r+1}{2}}\begin{bmatrix} n-1 \\ r \end{bmatrix}_q\right)^{k-1}}$$

if either $\mu = (n-2, 2)$ with $n \geq 5$ or $\mu = (2, 1^{n-2})$ with $n \geq 3$.

Behind the scenes

Technique

Theorem (Frobenius)

Let G be a finite group, let $k \in \mathbb{N}$, and, for each $i \in \{1, \ldots, k\}$, let A_i be a union of conjugacy classes in G. For any $g \in G$, the number of tuples $(t_1, \ldots, t_k) \in A_1 \times \cdots \times A_k$ such that $t_1 \cdots t_k = g$ is given by

$$\frac{1}{\#G} \sum_{\chi \in \mathrm{Irr}(G)} (\deg \chi)^{1-k} \chi(g^{-1}) \prod_{i=1}^{k} \sum_{t \in A_i} \chi(t).$$

For the symmetric group

Theorem (Murnaghan, Nakayama)

For all $n \in \mathbb{N}$ and $\lambda, \mu \vdash n$, we have

$$\chi^\lambda_\mu = \sum_{\substack{\text{border strip tab. } T \\ \text{of shape } \lambda \\ \text{and type } \mu}} (-1)^{ht\ T}.$$

en.wikipedia.org/wiki/Murnaghan–Nakayama_rule

- the set of squares filled with the integer *i* form a *border strip*, that is, a connected skew-shape with no 2x2-sq

The *height*, *ht*(T), is the sum of the heights of the border strips in *T*. The height of a border strip is one less than t

It follows from this theorem that the character values of a symmetric group are integers.

For some combinations of λ and ρ, there are no border-strip tableaux. In this case, there are no terms in the sum

Example [edit]

Consider the calculation of one of the character values for the symmetric group of order 8, when λ is the partition λ specifies that the tableau must have three rows, the first having 5 boxes, the second having 2 boxes, and the t tableau must be filled with three 1's, three 2's, one 3, and one 4. There are six such border-strip tableaux:

1	1	1	3	4
2	2			
2				

1	1	2	2	2
1	3			
4				

1	1	2	2	2
1	4			
3				

1	2	2	2	3
1	4			
1				

1	2	2	2	4
1	3			
1				

1	2	2	3	4
1	2			
1				

If we call these T_1, T_2, T_3, T_4, T_5, and T_6, then their heights are

$$\begin{aligned} ht(T_1) &= 0+1+0+0 = 1 \\ ht(T_2) &= 1+0+0+0 = 1 \\ ht(T_3) &= 1+0+0+0 = 1 \\ ht(T_4) &= 2+0+0+0 = 2 \\ ht(T_5) &= 2+0+0+0 = 2 \\ ht(T_6) &= 2+1+0+0 = 3 \end{aligned}$$

and the character value is therefore

$$\chi^{(5,2,1)}_{(3,3,1,1)} = (-1)^1 + (-1)^1 + (-1)^1 + (-1)^2 + (-1)^2 + (-1)^3 = -1-1-1+1+1-1 = -2$$

For $\mathrm{GL}_n\mathbb{F}_q$

Lemma (based on Green's work)

Suppose $n \in \mathbb{N}$, $d|n$, $\lambda \vdash n/d$, q is a prime power, $f \in \mathcal{F}_d(q)$, $\mu \vdash n$, $g \in \mathcal{T}_\mu^\square(q)$, and $h_1, \ldots, h_{\ell(\mu)}$ are the distinct irreducible factors of the characteristic polynomial of g. If some part of μ is not divisible by d, then $\chi^{f\mapsto\lambda}(g) = 0$. Otherwise, there exists $\tilde{\mu} \vdash n/d$ such that $\mu = d\tilde{\mu}$, and

$$\chi^{f\mapsto\lambda}(g) = (-1)^{\frac{n}{d}(d-1)} \chi^\lambda_{\tilde{\mu}} \prod_{i=1}^{\ell(\mu)} \frac{1}{\tilde{\mu}_i} \sum_{\substack{\beta_i \in \mathbb{F}_{q^{\mu_i}} \\ h_i(\beta_i)=0}} \theta(\beta_i)^{\ell_f[\tilde{\mu}_i]_{q^d}}.$$

Previous results on characters

Theorem (Steinberg)

For all $n \in \mathbb{N}$ *and* $\lambda, \mu \vdash n$, *if* $g \in \mathcal{T}_\mu^\square(q)$, *then* $\chi^{z-1 \mapsto \lambda}(g) = \chi_\mu^\lambda$.

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Proposition (Lewis-Reiner-Stanton)

For all $\chi \in \operatorname{Irr} \mathrm{GL}_n \mathbb{F}_q$ *and* $g \in \mathcal{T}_{(n)}(q)$, *if* $\chi(g) \neq 0$, *then* $\chi = \chi^{f \mapsto (n/d - r, 1^r)}$ *for some* f *with* $\deg f = d$ *and* $r \in \{0, \ldots, n/d - 1\}$.

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Compare to:

$$\chi^\lambda_{(n)} \neq 0 \implies \lambda = (n - r, 1^r) \text{ for some } r \in \{0, \dots, n-1\}.$$

Summary of proofs

- Proposition says only $\chi^{f \mapsto (n/d-r, 1^r)}$ are relevant.
- Lemma says only need to consider values of d dividing every part of μ.
- Plug character values into Frobenius' formula.
- Simplify. ☺

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- Plug character values into Frobenius' formula.
- Simplify. ☺

End up with a formula for

$$g_{k,\mu}(q) = \#\{(t_1, \ldots, t_k) \in \mathcal{T}_{(n)}(q)^k : t_1 \cdots t_k \in \mathcal{T}_\mu(q)\}.$$

Polynomiality

Corollary (to main result)

Suppose $n, k \in \mathbb{N}$ with $n > 2$. If $\mu \vdash n$ with $m_1(\mu) = 1$, then $g^{\square}_{k,\mu}(q)$ is a polynomial in q with rational coefficients.

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Suppose $n, k \in \mathbb{N}$ with $n > 2$. If $\mu \vdash n$ with $m_1(\mu) = 1$, then $g^{\square}_{k,\mu}(q)$ is a polynomial in q with rational coefficients.

Example

For all prime powers q,

$$g_{2,(2,1)}(q) = \frac{1}{18} q^6 (q-1)^7 (q+1)^3 (q^2+q+1).$$

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Example

For all prime powers q,

$$g_{3,(2,1)}(q) = \frac{1}{54} q^7 (q-1)^{10} (q+1)^6 (q^2 - q + 1)(q^2 + q + 1).$$

Corollary (to $g_{k,(n)}(q)$ formula)

Fix $n, k \in \mathbb{N}$. If n is prime, there exist degree-kn^2 polynomials $f_0, f_1, \ldots, f_{n-1} \in \mathbb{Q}[x]$ such that, for each $i \in \{0, \ldots, n-1\}$, we have

$$g_{k,(n)}(q) = f_i(q) \quad \text{for all prime powers } q \equiv i \pmod{n}.$$

Furthermore, $f_1 \neq f_0 = f_2 = f_3 = \cdots = f_{n-1}$.

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Example

If $n = k = 2$, then

$$f_0(q) = \tfrac{1}{8} q^3 (q-1)^3 (q^2 - 3q + 4),$$
$$f_1(q) = \tfrac{1}{8} q (q-1)^4 (q^3 - 2q^2 + 2q + 1).$$

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Example

If $n = 3$ and $k = 2$, then

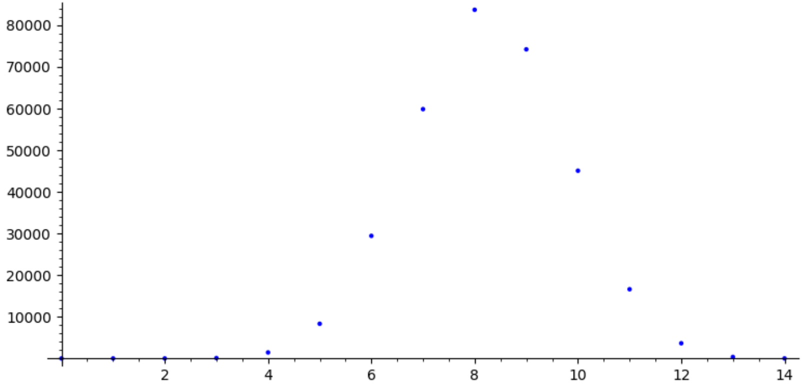
$$f_0(q) = \tfrac{1}{27} q^6 (q-1)^4 (q+1)^2 (q^6 - 4q^4 + 3q^3 + 5q^2 - 9q + 1),$$
$$f_1(q) = \tfrac{1}{27} q^3 (q-1)^5 (q+1) (q^9 + 2q^8 - 2q^7 - 3q^6 + 5q^5 + q^4 - 9q^3 - 4q^2 - 2q + 2).$$

Example

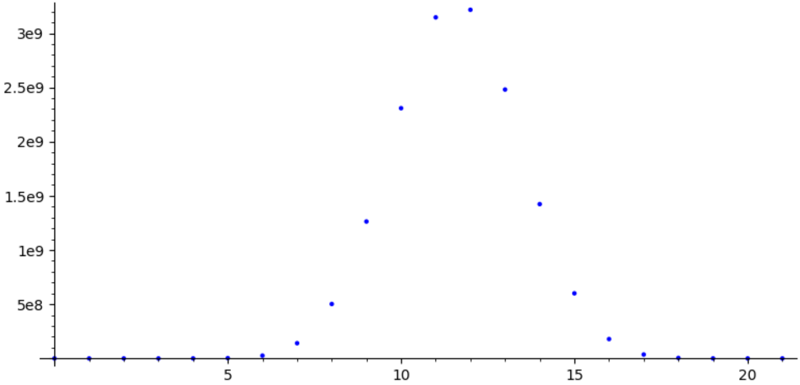
The generating functions for quasipolynomials are rational:

$$\sum_{q\geq 0} g_{2,(2)}(q)x^q = 2x^2(4x^{12} + 177x^{11} + 1821x^{10} + 8301x^9 + 22521x^8 + 37086x^7 + 41830x^6 + 29910x^5 + 14706x^4 + 4161x^3 + 717x^2 + 45x + 1)\Big/\left((1-x^2)^6(1-x)^3\right).$$

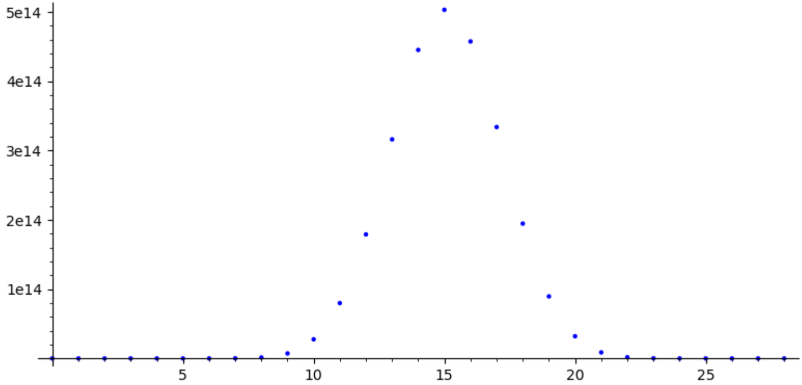
Coefficients of numerator of generating function of $g_{2,(2)}(q)$



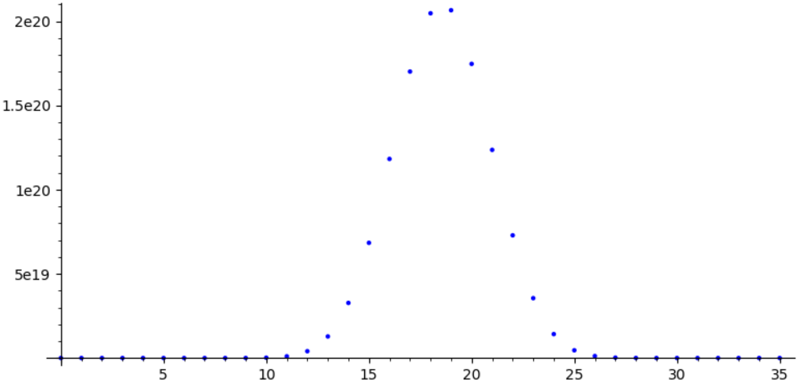
Coefficients of numerator of generating function of $g_{3,(2)}(q)$



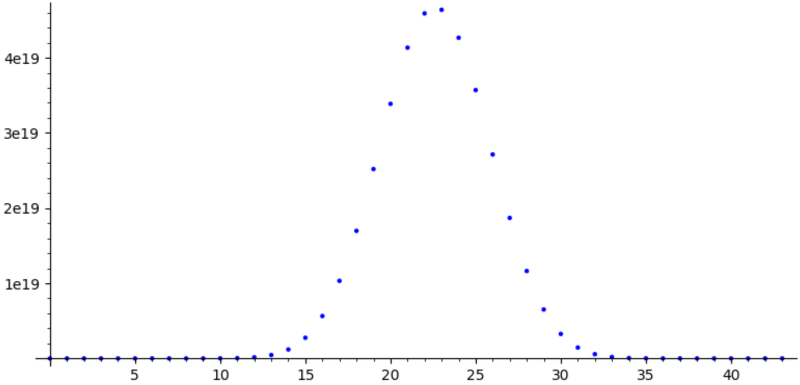
Coefficients of numerator of generating function of $g_{4,(2)}(q)$



Coefficients of numerator of generating function of $g_{5,(2)}(q)$



Coefficients of numerator of generating function of $g_{2,(3)}(q)$



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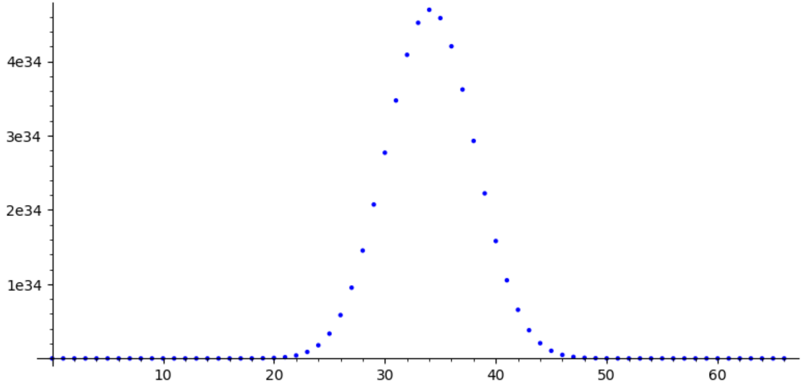

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- Refine the main factorization results to the level of conjugacy classes.
- Describe the numerator of $\sum_{q \geq 0} g_{k,\mu}(q) x^q$.

More open problems

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Thanks!